Exercise 5

Find the Laplace transform of the following expressions:

$$x + \sin x$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx.$$

Use the definition to find the answer.

$$\begin{aligned}\mathcal{L}\{x + \sin x\} &= \int_0^\infty e^{-sx}(x + \sin x)\, dx \\ &= \int_0^\infty (xe^{-sx} + e^{-sx}\sin x)\, dx \\ &= \int_0^\infty xe^{-sx}\, dx + \int_0^\infty e^{-sx}\sin x\, dx\end{aligned}$$

To avoid using integration by parts, write the first integrand as a derivative with respect to s and write the second integrand in terms of exponential functions by using Euler's formula.

$$\begin{aligned}&= \int_0^\infty \left(-\frac{\partial}{\partial s}\right) e^{-sx}\, dx + \int_0^\infty e^{-sx}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) dx \\ &= -\frac{d}{ds}\int_0^\infty e^{-sx}\, dx + \frac{1}{2i}\left[\int_0^\infty e^{(-s+i)x}\, dx - \int_0^\infty e^{-(s+i)x}\, dx\right]\end{aligned}$$

Evaluate each of the integrals.

$$\begin{aligned}&= -\frac{d}{ds}\left[\frac{1}{(-s)}e^{-sx}\right]\Bigg|_0^\infty + \frac{1}{2i}\left[\frac{1}{-s+i}e^{(-s+i)x}\Bigg|_0^\infty - \frac{1}{-s-i}e^{-(s+i)x}\Bigg|_0^\infty\right] \\ &= -\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) \\ &= \frac{1}{s^2} + \frac{1}{2i}\cdot\frac{s+i-s+i}{(s-i)(s+i)} \\ &= \frac{1}{s^2} + \frac{1}{\cancel{2i}}\cdot\frac{\cancel{2i}}{s^2+1}\end{aligned}$$

Therefore,

$$\mathcal{L}\{x + \sin x\} = \frac{1}{s^2} + \frac{1}{s^2+1}.$$